Digital Control Exam
Model Answer

Problem no. 1 part (a) :-

Stability check is solved using Juri test or bilinear transformation followed by Routh check. Pole locations method is for check only (i.e. Not Graded). In this exam it is considered to be 2/3 of the right answer

Given c/c eqn $\quad z^3 + 3z + 2 = 0$

① Juri test

- $P(1) = (1)^3 + 3(1) + 2 = 6 > 0$ 1st condition achieved
- $(-1)^n P(-1) = (-1)^3\left((-1)^3 + 3(-1) + 2\right) = 2 > 0$ 2nd condition achieved
- $|a_0| = 2 \;,\; |a_3| = 1 \longrightarrow a_0 > a_3$ doesn't achieve 3rd condition

The System is unstable

② Bilinear Transformation

c/c eqn $\quad z^3 + 3z + 2 = 0 \longrightarrow z = \frac{1+r}{1-r}$

$$\left(\frac{1+r}{1-r}\right)^3 + 3\left(\frac{1+r}{1-r}\right) + 2 = 0$$

$$(1+r)^3 + 3(1+r)(1-r)^2 + 2(1-r)^3 = 0$$

$$(1+r)(1+2r+r^2) + 3(1+r)(1-2r+r^2) + 2(1-r)(1-2r+r^2) = 0$$

$$1 + \cancel{3r} + \cancel{3r^2} + r^3 + 3 - \cancel{3r} - \cancel{3r^2} + 3r^3 + 2 - 6r + 6r^2 - 2r^3 = 0$$

$$2r^3 + 6r^2 - 6r + 6 = 0$$

r^3	2	-6
r^2	6	6
r^1	-8	
r^0	6	

Due to change in sign in the first column element the system is unstable

Problem no. 1 Part (b):-

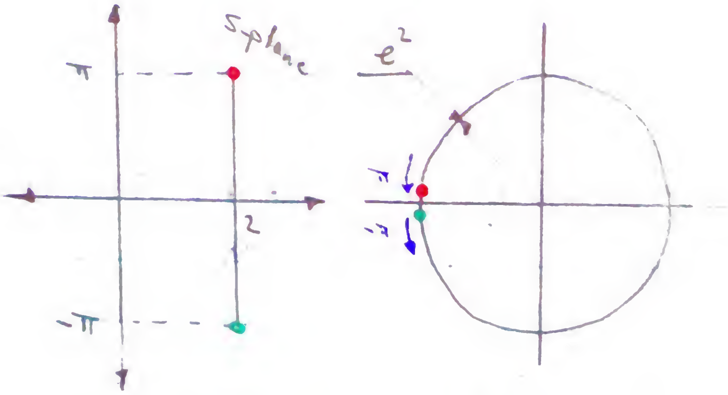
knowing that the relation between S-plane and Z-plane as follows

$$Z = e^{ST} \quad , \quad S = \sigma + j\omega$$

$$\therefore \quad Z = e^{(\sigma + j\omega)T}$$

$$= e^{\sigma T} e^{j\omega T}$$

$$= e^{\sigma T} \angle \omega T$$



for the given line Segment

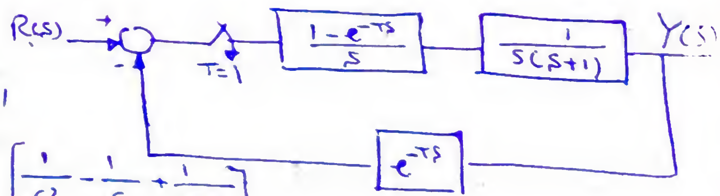
$$\sigma = 2 \quad , \quad \omega : -\pi \to \pi$$

$$Z = e^{2T} \angle \omega T = e^{2} \angle \omega \quad (T = 1)$$

The line Segment is mapped in Z-plane to a circle with Radius $r = e^{2} = 7.39$ starting at angle $= -\pi$ and ending at angle $= \pi$

Problem no. 2 Part (a):-

i) O.L.T.F $\quad \overline{GH}(z) = \mathcal{Z}\left[\dfrac{(1 - e^{-TS})(e^{-TS})}{S^2(S+1)}\right]_{T=1}$



$$\overline{GH}(z) = (1 - z^{-1})(z^{-1})\, \mathcal{Z}\left[\frac{1}{S^2(S+1)}\right] = \frac{z-1}{z^2}\, \mathcal{Z}\left[\frac{1}{S^2} - \frac{1}{S} + \frac{1}{S+1}\right]$$

$$= \frac{\cancel{z-1}}{z^{\cancel{2}}}\left(\frac{\cancel{z}}{(z-1)^{\cancel{2}}} - \frac{z}{z-1} + \frac{z}{z - e^{-1}}\right) = \frac{1}{z}\left(\frac{1}{z-1} - 1 + \frac{z-1}{z - 0.368}\right)$$

$$= \frac{(z - 0.368) - (z-1)(z - 0.368) + (z-1)^2}{z(z-1)(z - 0.368)} = \frac{z - 0.736 - z^2 + 1.368z + z^2 - 2z + 1}{z(z-1)(z - 0.368)}$$

$$\boxed{\overline{GH}(z) = \frac{0.368z + 0.264}{z(z-1)(z - 0.368)}}$$

ii) C.L.T.F $\quad \dfrac{Y(z)}{R(z)} = \dfrac{G(z)}{1 + \overline{GH}(z)}$

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-TS}}{S^2(S+1)}\right]_{T=1} = \frac{z-1}{z}\mathcal{Z}\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

$$= \frac{z-1}{z}\left[\frac{z}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-0.368}\right] = \frac{1}{z-1} - 1 + \frac{z-1}{z-0.368}$$

$$= \frac{0.368z + 0.264}{(z-1)(z-0.368)}$$

$$\therefore \frac{Y(z)}{R(z)} = \frac{(0.368z+0.264)/((z-1)(z-0.368))}{1+(0.368z+0.264)/(z(z-1)(z-0.368))} = \frac{z(0.368z+0.264)}{z(z-1)(z-0.368)+(0.368z+0.264)}$$

$$\boxed{\frac{Y(z)}{R(z)} = \frac{0.368z^2+0.264z}{z^3-1.368z^2+0.736z+0.264}}$$

iii) System Error Constants:

Position error constant $\quad K_p = \lim_{z\to1} \overline{GH}(z) = \lim_{z\to1}\frac{0.368z+0.264}{z(z-1)(z-0.368)} = \infty$

velocity error Constant $\quad K_v = \lim_{z\to1}(z-1)\,\overline{GH}(z) = \lim_{z\to1}\frac{(z-1)(0.368z+0.264)}{z(z-1)(z-0.368)} = 1$

acceleration error Constant $\quad K_a = \lim_{z\to1}(z-1)^2\,\overline{GH}(z) = \lim_{z\to1}\frac{(z-1)^2(0.368z+0.264)}{z(z-1)(z-0.368)} = 0$

iv) For unit step Input

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{\infty} = \text{Zero}$$

v) $y_{ss} = y(\infty) = r(\infty) - e_{ss}$

for unit step input $r(\infty) = 1$, $e_{ss} = 0$ (from the previous part)

$y_{ss} = 1 - 0 = 1$

Another Solution

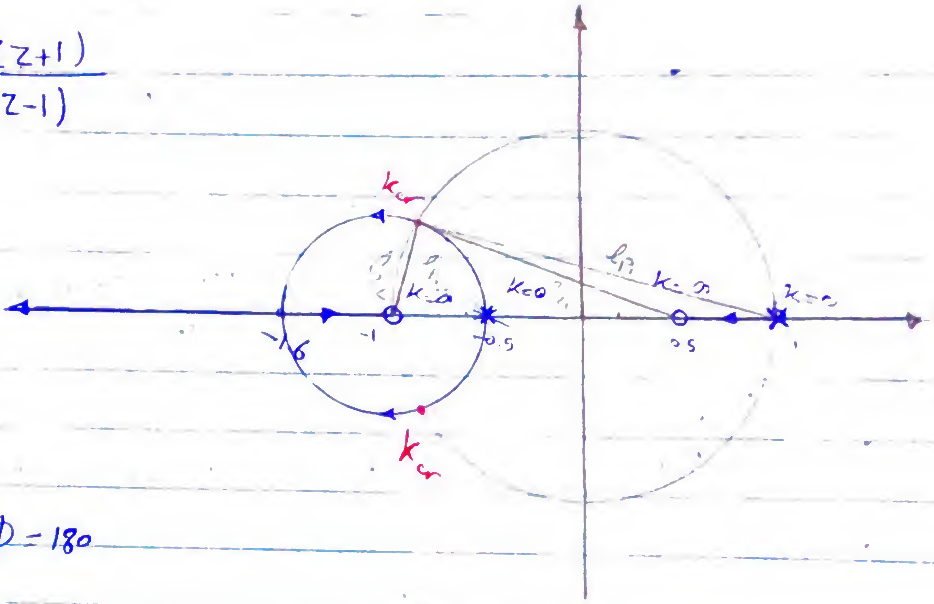
$$\frac{Y(z)}{R(z)} = \frac{0.368z^2+0.264z}{z^3-1.368z^2+0.736z+0.264}, \quad R(z) = \frac{z}{z-1} \text{ for unit step input}$$

$$Y(z) = \frac{0.368z^2+0.264z}{z^3-1.368z^2+0.736z+0.264}\cdot\frac{z}{z-1} \quad \text{Using final Value theory}$$

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{z\to1}\frac{z-1}{z}Y(z) = \lim_{z\to1}\frac{(z-1)(0.368z^2+0.264z)z}{z(z^3-1.368z^2+0.736z+0.264)(z-1)} = 1$$

Problem no. 2 Part (b) :-

$\because$ O.L.T.F $\quad H(z) = \dfrac{K(z-0.5)(z+1)}{(z+0.5)^2(z-1)}$



(1) $n_p = 3$ @ $z = 1, -0.5, -0.5$

$n_z = 2$ @ $z = 0.5, -1$

(2) Real part

$]-\infty, -1[\ ,\ ]0.5, 1[$

(3) Asymptotes : $n = n_p - n_z = 1$, $\phi = 180$

(4) Breaking Points : $1 + KGH = 0$

$K = \dfrac{-(z+0.5)^2(z-1)}{(z-0.5)(z+1)}$ — We search for K_{min} for breaking in

Z	-1.2	-1.3	-1.4	-1.5	-1.6	-1.7	-1.8	-1.9
K	3.171	2.726	2.558	2.5	2.497	2.525	2.572	2.631

Breaking away :- @ -0.5

Root Locus has eqn of Circle with D = 1.1 , Center at $(-1.05, 0)$

(5) Range of K for Stability :

the critical Stability occurs at the intersection between the root locus and the unit Circle with gain K_{cr}

$$K_{cr} = \frac{l_{p_1} l_{p_2} l_{p_3}}{l_{z_1} l_{z_2}} = \frac{1.92 * 0.61 * 0.61}{1.47 * 0.52} = 0.974 \text{ Approximate}$$

Range of K for stability $\boxed{0 < K < 0.974}$

Done